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Loxodromic Descent: A New Regularized Rotation Estimation

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Nomenclature

$\boldsymbol{C}_*$	=	variance–covariance matrix of $*$
$D*$	=	error of $*$
i	=	subscript for reference directions
k	=	subscript for measurement directions
ℓ	=	number of scalar measurements
$\boldsymbol{n}$	=	rotation axis
$\boldsymbol{P}_0$	=	coefficient matrix for the differential correction
$\boldsymbol{p}_{ki}$	=	vector of differential coefficients
$q_1, \ldots, q_4$	=	Rodrigues parameters
$\boldsymbol{R}$	=	rotation matrix
$\boldsymbol{u}$	=	measurement axis direction vector in the body system
$\boldsymbol{v}$	=	attitude reference direction vector in inertial space
w	=	scalar measurement
$\boldsymbol{w}$	=	measurement vector
α	=	right ascension of the rotation axis, rad
$\boldsymbol{\Delta e}(\Delta e_1, \Delta e_2, \Delta e_3)$	=	differential vector update
δ	=	declination or latitude of the rotation axis, rad
ρ	=	rhumb angle, rad
σ_ψ^2	=	variance of the global rotation error, rad^2
ϕ	=	rotation angle around the rotation axis, rad
Ψ	=	global error rotation vector
ψ_i	=	rotation error angle component, rad
$(*)^0$	=	arc $(*)$, deg
$[[*]]$	=	skew symmetric vector product matrix operator for vector $*$

Introduction

THE rotation linking the body reference system to a representation of its orientation in inertial space is commonly called the attitude of this vehicle. Considering satellites and their attitude control, attitude is normally described by a full-blown rotation, mathematically represented by a (3×3) orthogonal matrix of determinant one. This matrix can be parametrized in many ways by three or four parameters, as described by Shuster [1]. But the minimum comprises three parameters, as was shown by Stuelpnagel [2]. It is well known that any three such conventional rotation parameters define a 3-D rotation uniquely. But in the opposite direction this is not true. On the contrary! For any of these reference systems, there are isolated rotations that correspond to an infinity of rotation parameters of a given parameterization. These singular points are different for different parameterizations, and estimations in their vicinity are known to be subject to major numerical problems. This is the reason why Euler angles, for example, normally disqualify in automatic control algorithms. One could fall back on the four singularity-free Rodrigues parameters [3], but typically in extended Kalman filters the interdependence of the the Rodrigues parameters forces one to nevertheless work with a three-parameter parameterization [4]. An extended discussion of the problem is given by Markley [5], who recently proposed what he called an orthogonal filter [6] bypassing the problems of constraints and singularities. This filter is more involved than an extended Kalman filter and in the end leaves the common question unaltered, whether the initial convergence is always ensured in practice even if starting with reasonable initial conditions.

The purpose of this paper is to describe an alternative way out, consisting of a three-parameter parameterization. These parameters are combined with a particular form of their differential increments, which regularizes the matrix algebra of attitude estimation everywhere on (the rotation group) SO(3). The shape of these increments allows then to implement a descent method that converges everywhere by employing the rhumb line technique. A rhumb line or loxodrome is an arc on a 3-D sphere (S^2) that makes a constant angle with all meridians linked to a fixed pole. Therefore it corresponds to a straight line on a corresponding Mercator projection. Loxodromes have been employed until recently for navigation on the oceans and are still in use for thruster-supported slew maneuvers of spin-stabilized satellites ([7], pp. 651–654). As SO(3) maps onto S^3, one will need to apply a rhumb line twice in cascade, each time on a S^2 subspace, to achieve a consistent correction step in all three parameters respecting only the step length of the robust parameter together with the direction suggested by an estimated raw differential correction vector. It is proposed to call this method the "loxodromic descent" and it could be used as a rather simple building block in an extended Kalman filter.

The loxodromic descent is especially conceived for application to scalar attitude measurements, thus also for cases where none or not all measurements one disposes of allow the construction of vector measurements required by the Wahba-based [8] estimation methods. But even if all measurements can be transformed into vector measurements, a recent study [9] demonstrates that there are scalar-based methods, to which the loxodromic descent applies, that outperform the accuracy of vector-based estimation schemes by up to 1 order of magnitude in particular but not exceptional circumstances. The test of the descent method itself, which is presented at the end of this note, consists in verifying that similar measurements lead to the same (optimal) attitude estimates, independently of the reference system used, or in other words equally well at the regularized singularities as remotely from such attitudes. This test employs data referring to a realistic scenario with six observed directions, where two of them cannot be reconstructed in the body system due to lack of

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complete measurement coverage. Estimation accuracy on its own is not the subject of the test, because one only verifies that the accuracy achieved away from the singularities, studied in the paper by Fraiture [9], is also obtained at the regularized singularities.

In the next section the typical known scalar measurement types are recalled. Thereafter, a largely conventional estimation scheme is presented, in which the algebraic regularization of the coefficient matrices is achieved. The differential correction vector obtained in this way is the input to the novel descent method. The background of the strategy employed to test the descent method is described in the last but one section. Ultimately, simulation results are given that demonstrate the numerical effectiveness of the loxodromic descent.

Measurement Models

One requires a minimum of three scalar projection measurements to be able at all to determine a 3-D rotation. This condition is not sufficient. It is further necessary that these measurements are generated by at least two noncollinear measurement directions $\boldsymbol{u}_k$ known in the body system and equally two observed noncollinear reference directions $\boldsymbol{v}_i$ defined in inertial space. It is implicitly assumed that direction vectors are normalized. With this minimum configuration four combinations of projection measurements of the form

$$(\boldsymbol{u}_k)'\boldsymbol{R}\boldsymbol{v}_i = w_{ki} \tag{1}$$

can be obtained. In (1) the accent denotes transposition and $\boldsymbol{R}$ is assumed to be the rotation matrix transforming inertial coordinates to body coordinates. The solution of minimum sets of projection measurements have been studied [10] and it was shown that solutions are then normally not unique. Here, however, one assumes that there are definitely more than a minimum of measurements to avoid any complications due to such ambiguities. Projection measurements typically apply to magnetometer readings and to interferometric phase differences of carrier signals of satellites of the general positioning system (GPS) as explained by Fraiture [9]. The directions $\boldsymbol{u}_k$ may represent antenna baseline directions on a space vehicle, and $\boldsymbol{v}_i$ are then the directions of GPS satellites. One observes that

$$w_{ki} = \boldsymbol{u}_k'\boldsymbol{R}_a'(\boldsymbol{R}_a\boldsymbol{R}\boldsymbol{R}_b')\boldsymbol{R}_b\boldsymbol{v}_i \tag{2}$$

This equality demonstrates that w_{ki} is invariant under an asymmetric similarity transformation involving two independent valid and arbitrary rotation matrices $\boldsymbol{R}_a$ and $\boldsymbol{R}_b$ applied to the measurement and reference vector spaces, respectively. Attention is drawn to this rather obvious property, because it enables tests with the same measurements but with different unknown rotation matrices $(\boldsymbol{R}_a\boldsymbol{R}\boldsymbol{R}_b')$ to be estimated. Incidentally, this similarity principle also constitutes the basis for the sequential rotation algorithm included in the QUEST method [11] in a quite different context. In the present note it is employed to artificially move the simulated unknown rotation $\boldsymbol{R}$ in (1) onto or very close to the regularized singularities.

There is a second equally important measurement model representing measurements of projection ratios, namely,

$$\frac{(\boldsymbol{u}_{k1})'\boldsymbol{R}\boldsymbol{v}_{i1}}{(\boldsymbol{u}_{k2})'\boldsymbol{R}\boldsymbol{v}_{i2}} = w_{k1,i1/k2,i2} \tag{3}$$

Such measurements can be applied to star tracker data as explained by Habermann [12] and implemented in ESA operational software for the ground control of astronomical observatory satellites. The invariance mentioned before remains valid here as well. Caveat! The measurement Eq. (3) leaves the value of the determinant of $\boldsymbol{R}$ undetermined. Consequently, if only projection ratios are involved, one has either to use a parameterization that implies the unit determinant, for example, any Euler angles, or to include an explicit constraint if a nontrigonometric parameterization is employed, for example, the Rodrigues parameters. The latter parameters were also called Euler's parameters by Gray [13] and are denoted by "Euler symmetric parameters" in the book edited by Wertz ([7], pp. 414–416), before their true author was traced back [14] a few decades ago and Rodrigues's contribution was compared with Euler's original work [15].

Resolution Scheme

In this section one will achieve the algebraic regularization of the equation for the differential correction of the rotation estimate based on (1). To start with, the Euler rotation parameters are introduced. They embody Euler's theorem stating that a rotation consists of a rotation axis $\boldsymbol{n}$ and a rotation angle ϕ around this axis. Basic information can be found in Wertz ([7], p. 413) about the well-known expansion of $\boldsymbol{R}$ due to Euler himself and based on ϕ and $\boldsymbol{n}$, namely,

$$\boldsymbol{R} = \cos\phi\boldsymbol{I}_3 + (1-\cos\phi)\boldsymbol{n}\boldsymbol{n}' - \sin\phi[[\boldsymbol{n}]] \tag{4}$$

where $\boldsymbol{I}_3$ is a unit matrix and the double square brackets represent a skew-symmetric matrix equivalent to a vector product matrix operator, or

$$[[\boldsymbol{n}]] = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

By writing $\boldsymbol{n}$ in polar coordinates as $n_1 = \cos\alpha\cos\delta$, $n_2 = \sin\alpha\cos\delta$, and $n_3 = \sin\delta$, the Euler rotation parameters reduce to three scalar parameters, namely, α, δ, and ϕ. Obviously not only $\delta = \pm\pi/2$ are singular points of this parameterization, but also $\phi = 0$ leaves $\boldsymbol{n}$ undefined. The latter singularity cannot be handled by the loxodromic descent as will become obvious hereafter. To circumvent this problem, one has to have recourse to the Rodrigues parameters, namely,

$$q_1 = n_1\sin\phi/2 \qquad q_2 = n_2\sin\phi/2 \qquad q_3 = n_3\sin\phi/2$$
$$q_4 = \cos\phi/2$$

where one substitutes $\pi/2 - \phi/2$ by the angle γ leading to

$$q_1 = \cos\alpha\cos\delta\cos\gamma \qquad q_2 = \sin\alpha\cos\delta\cos\gamma$$
$$q_3 = \sin\delta\cos\gamma \qquad q_4 = \sin\gamma$$

which has singular points at $\gamma = \pm\pi/2$ as well as at $\delta = \pm\pi/2$. But the factors being the sources of these singularities can be isolated, because

$$\begin{aligned} \frac{\partial(\boldsymbol{n}\cos\gamma)}{\partial\alpha} &= \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}\cos\gamma\cos\delta \\ \frac{\partial(\boldsymbol{n}\cos\gamma)}{\partial\delta} &= \begin{pmatrix} -\cos\alpha\sin\delta \\ -\sin\alpha\sin\delta \\ \cos\delta \end{pmatrix}\cos\gamma \end{aligned} \tag{5}$$

The differential of $(\boldsymbol{u}_k)'\boldsymbol{R}\boldsymbol{v}_i$ can thus be rearranged to obtain

$$\mathrm{d}\,[(\boldsymbol{u}_k)'\boldsymbol{R}\boldsymbol{v}_i] = \mathrm{d}[w_{ki}(\boldsymbol{R})] = \boldsymbol{p}_{ki}'\begin{pmatrix} \mathrm{d}\gamma \\ \cos\gamma\,\mathrm{d}\delta \\ \cos\delta\cos\gamma\,\mathrm{d}\alpha \end{pmatrix} = \boldsymbol{p}_{ki}'\,\mathbf{d}\boldsymbol{e} \tag{6}$$

where $w_{ki}(\boldsymbol{R})$ denotes a scalar function of the rotation parameters of $\boldsymbol{R}$, whereas the vector $\boldsymbol{p}_{ki}$ is defined by

$$\boldsymbol{p}'_{ki} = \left|\frac{\partial w_{ki}(\boldsymbol{R})}{\partial\gamma}, \frac{1}{\cos\gamma}\frac{\partial w_{ki}(\boldsymbol{R})}{\partial\delta}, \frac{1}{\cos\delta\cos\gamma}\frac{\partial w_{ki}(\boldsymbol{R})}{\partial\alpha}\right|$$

Combining the $3 < \ell$ available measurements in the vector $\boldsymbol{w}$ and employing the subscript zero to denote a quantity at the linearization origin yields

$$\boldsymbol{w} \approx \boldsymbol{w}(\boldsymbol{R}_0) + \boldsymbol{P}_0\boldsymbol{\Delta}\boldsymbol{e} \tag{7}$$

This vector equation is the conventional start point of a steepest descent step, where $\mathbf{d}\boldsymbol{e}$ in (6) now becomes an everywhere single valued and finite unknown increment $\mathbf{\Delta e}' = |\Delta\gamma, \cos\gamma\Delta\delta, \cos\delta\cos\gamma\Delta\alpha|$. The rows of the $\ell \times 3$ coefficient matrix $\boldsymbol{P}_0$ contain the ℓ vectors $\boldsymbol{p}'_{ki}$. A similar derivation applies to (3) leading also to (7) with an adapted matrix $\boldsymbol{P}_0$. The advantage is then that the implicit norm indetermination of the Rodrigues vector is lifted by the introduction of γ, δ, and α. Turning back to the general case and putting $\cos\gamma$ and $\cos\delta\cos\gamma$ in evidence, $\boldsymbol{P}_0$ is no longer singular when $\delta = \pm\pi/2$ and/or $\gamma = \pm\pi/2$. And $\boldsymbol{P}_0$ is regular everywhere else if the ℓ measurements result from at least two noncollinear measurement axes and reference directions. Consequently, the linearization of $\boldsymbol{R}$ has been regularized. And because $3 < \ell$ is assumed, there is overdetermination and the computation of $\mathbf{\Delta e}$ in a steepest descent iteration step is based on the equation

$$(\boldsymbol{P}'_0\boldsymbol{C}_w^{-1}\boldsymbol{P}_0)\mathbf{\Delta e} = (\boldsymbol{P}'_0\boldsymbol{C}_w^{-1})[\boldsymbol{w} - \boldsymbol{w}(\boldsymbol{R}_0)] \tag{8}$$

in a shape required by the Gauss–Markov theorem [16] leading to a minimum variance estimate. If $E(*)$ stands for the expectation of $*$, one has

$$\boldsymbol{C}_w = E(\boldsymbol{Dw}\boldsymbol{Dw}'), \qquad \boldsymbol{C}_e = E(\mathbf{\Delta e}\mathbf{\Delta e}') = (\boldsymbol{P}'_0\boldsymbol{C}_w^{-1}\boldsymbol{P}_0)^{-1} \tag{9}$$

where the matrix $\boldsymbol{C}_w$ is the $(\ell \times \ell)$ (nonsingular) error covariance matrix of the measurements and $\boldsymbol{C}_e$ is the (3×3) covariance matrix of the direction error $\mathbf{d}\boldsymbol{s}$ radian of the Rodrigues unit vector. Such a direction error satisfies the differential $\mathbf{d}\boldsymbol{s}^2$ which is equal to

$$\mathbf{d}\,\boldsymbol{s}^2 = \mathrm{d}\gamma^2 + \cos^2\gamma\,\mathrm{d}\delta^2 + \cos^2\delta\cos^2\gamma\,\mathrm{d}\alpha^2 \tag{10}$$

and more specifically

$$\boldsymbol{C}_e = \begin{pmatrix} \sigma_{\gamma\gamma} & \cos\gamma_0\sigma_{\gamma,\delta} & \cos\gamma_0\cos\delta_0\sigma_{\gamma,\alpha} \\ \cos\gamma_0\sigma_{\gamma,\delta} & \cos^2\gamma_0\sigma_{\delta\delta} & \cos^2\gamma_0\cos\delta_0\sigma_{\delta,\alpha} \\ \cos\gamma_0\cos\delta_0\sigma_{\gamma,\alpha} & \cos^2\gamma_0\cos\delta_0\sigma_{\delta,\alpha} & \cos^2\gamma_0\cos^2\delta_0\sigma_{\alpha\alpha} \end{pmatrix} \tag{11}$$

where $\sigma_{ab} = E(\Delta a\Delta b)$, which is the covariance of the errors on the angles a and b.

Loxodromic Descent

At each step of the steepest descent, $\boldsymbol{P}_0$ needs to be updated with the improved values of γ, δ, and α. For the update of δ one can first compute the new angle γ to obtain $\Delta\delta$ simply by dividing Δe_2 by $\cos\gamma$, provided γ is not close to $\pm\pi/2$. If γ is close to $\pm\pi/2$ one could believe that a clever size constraint on $\Delta\delta$ may force convergence, as described by Fraiture [17]. Unfortunately, numerical simulations show that the major drawback of this procedure is the resulting limitation of the achievable convergence near the regularized singularity, the corresponding accuracy loss, and the necessity of finding out how to define the convergence limit accordingly. But it appears that the geometry of an arc increment forced to go along a loxodrome or rhumb line in a Mercator projection is a valid, accurate, and feasible alternative. The descent step is then broken down in a first part considering a Mercator projection with the latitude γ and the longitude δ. After having performed the step on that projection, a second Mercator projection is considered where this time the latitude is δ and the longitude is α. One thus considers a descent step starting at the value γ_0 on the meridian on which γ is defined. Combined with the rhumb angle this fixes a consistent step length on the small circle of δ orthogonal to the meridian by applying the constraint that the path followed is a rhumb line. In the first part one computes the value of $\Delta\delta$ consistent with $\Delta\gamma$ and a rhumb angle denoted by ρ_0. A similar procedure will be applied to derive a consistent value of $\Delta\alpha$ based on the result of the first step.

Let one first consider the unit sphere (S^2) with the (north) pole at $\gamma = \pi/2$ and note that $|\gamma| \le \pi/2$ by definition. Although $|\delta| \le \pi/2$ on S^3 where the Rodrigues parameters are defined, there is no principal problem to allow $-\pi < \delta \le \pi$, should one not need δ in a second loxodrome where it needs to be confined inside the interval $(-\pi/2, \pi/2)$. Therefore, one shall exploit the natural ambiguities available when translating the Rodrigues parameters of the initial

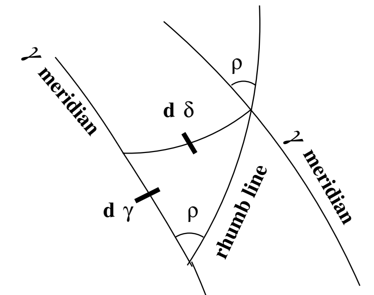

Fig. 1 The loxodrome geometry.

rotation and select δ to be in the good interval and keep it there in the subsequent iteration steps.

When computing the differential correction $\mathbf{\Delta e}$, one starts from the point $(\gamma_0, \delta_0, \alpha_0)$ being the origin for the potential correction keeping α_0 constant. This is pictorially shown in Fig. 1. Assuming plane geometry, the rhumb angle ρ_0 is defined by

$$\cot\rho_0 = \Delta e_2/\Delta e_1 = \cos\gamma_0\Delta\delta/\Delta\gamma \tag{12}$$

If one requires that the start rhumb angle stays constant all over the path covered by a descent step, (12) has to apply infinitesimally everywhere on that path, or

$$\mathrm{d}\,\delta = \cot\rho_0\,\mathrm{d}\gamma/\cos\gamma \tag{13}$$

The length of a corresponding differential line element on the unit sphere is $(\mathrm{d}\gamma^2 + \cos^2\gamma\,\mathrm{d}\delta^2)$. Hence, the metric weight of the arc length of $\mathrm{d}\gamma$ is one. Therefore there is no reason to mistrust the value obtained for $\Delta\gamma$ except if $\pi/2 < |\gamma_0 + \Delta\gamma|$, because then one has left the definition interval of γ, creating a major definition ambiguity for δ. To overcome this problem one can multiply $\Delta\gamma$ by a gradually decreasing factor f up to the point where

$$|\gamma| = |\gamma_0 + f\Delta\gamma| < \pi/2$$

Now, both the step length for the γ update is known as well as the rhumb angle for applying the step. The corresponding value of $\Delta\delta$, taking the narrowing of the latitude lines between meridians as a function of γ into account, is determined by the rhumb line integral:

$$\Delta\delta = \cot\rho_0 \int_{\gamma_0}^{\gamma} \frac{\mathrm{d}\gamma}{\cos\gamma} = \cot\rho_0\{\ell n\tan[0.5(\gamma + 0.5\pi)] - \ell n\tan[0.5(\gamma_0 + 0.5\pi)]\} \tag{14}$$

It may happen that the incremented value of δ suggested by (14) is larger than $\pi/2$ in absolute value. As mentioned before this leads to problems in the determination of $\Delta\alpha$ later on. Therefore one selects a value of the δ correction $|\Delta\delta_{\mathrm{red}}| < |\Delta\delta|$ such that $|\delta_0 + \Delta\delta_{\mathrm{red}}| < \pi/2$. To stay consistent also the γ increment must be adapted and the incremented value γ_{red} is obtained by straightforwardly solving

$$\Delta\delta_{\mathrm{red}} = \cot\rho_0 \int_{\gamma_0}^{\gamma_{\mathrm{red}}} \frac{\mathrm{d}\gamma}{\cos\gamma}$$

analytically. It may happen that the differential increment $\mathbf{\Delta e}$ becomes exactly zero after a few iterations, which means that one had the solution already before applying the last step. If one erroneously proceeds with the iteration at this point, applying (12) will result in an arithmetic exception in the software execution. Of course, using pieces of loxodromes works equally well when $|\gamma|$ is far from $\pm\pi/2$.

In the same descent step one now has to find an acceptable value for $\Delta\alpha$ by again making a plane approximation now involving the δ meridians, with respect to which the rhumb angle ρ_1 is given by

$$\cot\rho_1 = \frac{\cos\gamma_0\cos\delta_0\Delta\alpha}{\cos\gamma_0\Delta\delta} \tag{15}$$

The rhumb line integral then yields

$$\Delta\alpha = \cot\rho_1\{\ell n \tan[0.5(\delta + 0.5\pi)] - \ell n \tan[0.5(\delta_0 + 0.5\pi)]\} \tag{16}$$

which is the consistent α increment. The procedure given before applies once more with the exception that α can vary from $-\pi$ to $+\pi$. To avoid numerical problems it is indicated to each time cutoff the updated α values at $+$ or $-\pi$.

The whole procedure is the *loxodromic descent*.

Sequential Estimation

The ability to update an estimate on the basis of one new measurement only is of paramount importance for online control and more in particular for the application of Kalman filtering. Assume that an estimate of the rotation is available in the form of the augmented polar coordinates $[\gamma_m, (\cos\gamma_m)\delta_m, (\cos\gamma_m\cos\delta_m)\alpha_m]$ with a covariance matrix $\boldsymbol{C_e}$ defined in (9) and (11). This means that three trivial measurements equations $\gamma = \gamma_m$, $(\cos\gamma)\delta = (\cos\gamma_m)\delta_m$, and $(\cos\gamma\cos\delta)\alpha = (\cos\gamma_m\cos\delta_m)\alpha_m$ are given with their covariance matrix at the start. If one supplementary (uncorrelated) measurement w_s with variance σ_s^2 is added, (6) describes the resulting measurement equation, yielding here the vector $\boldsymbol{p}_{s0}$. The redundant system of linearized measurement equations reads

$$\begin{vmatrix} \boldsymbol{I}_3 \\ \boldsymbol{p}'_{s0} \end{vmatrix} \boldsymbol{\Delta e} = \begin{vmatrix} \boldsymbol{0_3} \\ w_s - w_{s0} \end{vmatrix} \tag{17}$$

where $\boldsymbol{\Delta e}$ is defined in (5) with γ and δ equal to γ_m and δ_m in the first step of the loxodromic descent and updated for each descent step again. The measurement covariance matrix is

$$\boldsymbol{C}_{\text{tot}} = \begin{vmatrix} \boldsymbol{C_e} & \boldsymbol{0_3} \\ \boldsymbol{0'_3} & \sigma_s^2 \end{vmatrix}$$

where $\boldsymbol{0_3}$ is a 3-D null vector. When applying the loxodromic descent to a Gauss–Markov-based sequential estimation step, the estimate as a function of the estimated differential increment is handled in the same way as described in the preceding section. At each iteration step this yields a new value for $[\gamma_m, (\cos\gamma_m)\delta_m, (\cos\gamma_m\cos\delta_m)\alpha_m]$. This procedure can easily be adapted to any number of additional measurements that one wants to incorporate at once.

Basic Accuracy Considerations

To assess the numerical performance of the loxodromic descent, one will compare the rms estimation errors based on identical measurements once in a reference system in which the estimate is far from the singularities and the two cases in which the estimation is moved to the immediate vicinity of the γ and δ singularity separately. To show that accuracies are not affected by the relative location of the singularity, one has to enforce reference independence based on the following considerations.

For the estimation errors themselves we dispose of the exact attitude solution, because one is dealing with a simulation. Hence, one will compute the actual error for each simulated trial. It is therefore proposed to approximate the error of an estimated 3-D rotation $\boldsymbol{R}$ by a small rotation $\boldsymbol{R}_e$ called the error rotation matrix. Thus if $\boldsymbol{R}_0$ is the error-free rotation for which a valid but error-corrupted $\boldsymbol{R}$ exists, one has

$$\boldsymbol{R}\,\boldsymbol{R}'_0 = \boldsymbol{R}_e \approx \boldsymbol{I}_3 + \begin{vmatrix} 0 & D\psi_3 & -D\psi_2 \\ -D\psi_3 & 0 & D\psi_1 \\ D\psi_2 & -D\psi_1 & 0 \end{vmatrix} = \boldsymbol{I}_3 + [[\boldsymbol{D\Psi}]]$$

where $\boldsymbol{D\Psi}' = |D\psi_1, D\psi_2, D\psi_3|$ are the error rotation angles and $D\psi_0^2 = |\boldsymbol{D\Psi}|^2$ is the square of the global rotation error by definition, as was introduced by Fraiture [17]. The global rotation error $D\psi_0$ is a small angle whose exact value can in fact be found from

$$\text{Tr}\,(\boldsymbol{R}\boldsymbol{R}'_0) = 1 + 2\cos(D\psi_0) \tag{18}$$

and this is the formula applied in the tests.

One can now verify the claim that the global rotation error is not affected by asymmetric similarity transformations having a shape as presented in (2). To this aim it is sufficient to show that $\text{Tr}(\boldsymbol{R}\boldsymbol{R}'_0)$ is invariant under such a transformation. Applying the asymmetric similarity transformation to $\boldsymbol{R}$ and $\boldsymbol{R}_0$ separately, $\boldsymbol{R}_e$ yields

$$\text{Tr}[(\boldsymbol{R}_a\boldsymbol{R}\boldsymbol{R}_b)(\boldsymbol{R}_a\boldsymbol{R}_0\boldsymbol{R}_b)'] = \text{Tr}[(\boldsymbol{R}_a\boldsymbol{R}\boldsymbol{R}_b)(\boldsymbol{R}'_b\boldsymbol{R}'_0\boldsymbol{R}'_a)]$$
$$= \text{Tr}(\boldsymbol{R}_a\boldsymbol{R}\boldsymbol{R}'_0\boldsymbol{R}'_a) = \text{Tr}(\boldsymbol{R}\boldsymbol{R}'_0\boldsymbol{R}'_a\boldsymbol{R}_a) = \text{Tr}(\boldsymbol{R}\boldsymbol{R}'_0)$$

This argument also holds for projection measurements defined in (3) and it can trivially be generalized to vector equations involving vector measurements as defined in Wahba's problem [8].

Numerical Verification

For the numerical verifications a simple GPS attitude scenario is imagined. It comprises two baseline vectors $\boldsymbol{u}$ lying in different planes intersecting on the x axis. These baselines observe up to six reference directions corresponding to GPS satellites, which are not all simultaneously visible in the different hemispheres (which point mainly into the $+z$ direction) of the two baselines. Right ascension and declination expressed in degrees is employed to define all these directions. For the baselines one has $\boldsymbol{u}_1(90.0, -30.0) = |\cos 90\cos 30, \sin 90\cos 30, -\sin 30|$ and $\boldsymbol{u}_2(270.0, -30.0)$. Similarly the six reference vectors are arbitrarily set to

$$\boldsymbol{v}_1(47.0, -10.0), \quad \boldsymbol{v}_2(65.0, 20.0), \quad \boldsymbol{v}_3(181.0, 63.0)$$
$$\boldsymbol{v}_4(142.0, 41.0), \quad \boldsymbol{v}_5(222.0, -3.0), \quad \boldsymbol{v}_6(313.0, 57.0)$$

All the measurements with baseline visibility at 2^0 from grazing incidence are employed (in fact nine in this case). Moreover, all measurements are subjected to bias-free uncorrelated uniformly distributed random errors within the interval $(-0.005, +0.005)$. Let $\boldsymbol{R}_x(a)$ represent a rotation of a degrees around the x axis. Three test rotations $\boldsymbol{R}_0$ have been studied, namely,

$$\boldsymbol{R}_{01} = \boldsymbol{R}_x(0.10)\boldsymbol{R}_y(0.05)\boldsymbol{R}_z(0.28)$$
$$\boldsymbol{R}_{02} = \boldsymbol{R}_x(0.10)\boldsymbol{R}_y(0.05)\boldsymbol{R}_z(28.0)$$
$$\boldsymbol{R}_{03} = \boldsymbol{R}_x(30.0)\boldsymbol{R}_y(40.0)\boldsymbol{R}_z(50.0)$$

As shown in Table 1, the first rotation is in the immediate vicinity of the γ singularity (at 0.15^0), whereas the second is very near the δ singularity (at 0.27^0). The third case is far from any singularity. To ensure estimation equivalence not only the measurements, but also the random error corruption of these measurements have to be the same for the three samples employed to estimate any of the three rotations $\boldsymbol{R}_{0j}$. To achieve equality of measurements, the $\boldsymbol{R}_a$ in the asymmetric similarity in (2) is found from $\boldsymbol{R}_a = \boldsymbol{R}_{0j}\boldsymbol{R}'_{01}$ where $j = 1$, 2, or 3. The initial condition to start the loxodromic descent algorithm is in all trials set to $\boldsymbol{R}_{\text{init}} = \boldsymbol{R}_x(5.5)\boldsymbol{R}_y(5.5)\boldsymbol{R}_z(5.5)\boldsymbol{R}_{0j}$ corresponding to an initial global rotation error of 9.67^0. In all trials the iteration end is subject to the condition $(\Delta e_1^2 + \Delta e_2^2 + \Delta e_3^2)^{1/2} < 1.10^{-7}$ deg.

As expected an equal rms global rotation error of 0.05096197 deg was found for all three samples of 10,000 estimations each. The loxodromic descent functioned flawlessly in all 30,000 trials discussed here as well as in all other trials attempted so far, demonstrating reliability. Nevertheless, some minor differences in the numerical performance were observed. They concern the mean number of iterations (ITER. NR. in Table 1) required and the reported maximum number (ITER. MAX) of iterations observed in a sample. These numbers are displayed in Table 1 together with the angles γ, δ,

Table 1 Test rotations and statistical iteration performances

	γ	δ	α	ITER. NR.	ITER. MAX
$\boldsymbol{R}_{01}$	**89.868**	65.026	26.445	7.146	9
$\boldsymbol{R}_{02}$	78.000	**89.731**	14.565	7.856	14
$\boldsymbol{R}_{03}$	51.741	48.509	28.640	5.109	6

and α applicable to the error-free test rotations. These data show that in the mean, two to three supplementary iterations may be required when operating close to the regularized singularities. Further samples not reported here indicate that a higher precision of the initial conditions has almost no influence on the number of required iterations, demonstrating robustness.

Conclusions

A scalar estimation of the 3-D rotation by means of a novel descent method has been tested. The parameterization employed consists of a transformation of the Rodrigues parameters into three polar coordinates, which allows to move the critical factors, which normally lead to singular Jacobian matrices in differential correction, into the differential correction vector. The algebra of the differential correction being regularized in this way only leaves a convergence problem at the regularized parameterization singularities. Also this problem could be circumvented by driving the differential correction steps along loxodromes. An extensive computer simulation has been performed, comparing the estimation of a well-behaved test rotation with test rotations at the regularized singularities. The estimation results obtained are equal in all cases. The only difference consists of a slightly increased need of iteration steps when the estimate is located at or in the vicinity of a regularized singularity. Each sample consists of 10,000 trials differing by realistic errors corrupting the error-free measurements starting the iteration some 10 deg away from the optimal estimate in each case. In the 30,000 trials that are discussed in this note and in all other cases tested, no flaws have occurred. One can thus conclude that the loxodromic descent is effective, reliable, and robust over the whole range of the three polar parameters covering all valid three dimensional rotations.

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